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Stability of developing flow in a pipe: non-axisymmetric disturbances

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Spatial stability results for the developing flow in a rigid circular pipe are presented for the velocity profile obtained by the Hornbeck (1963) method and compared with the available temporal stability results for the velocity profile obtained by Sparrow, Lin & Lundgren (1964). The disturbance is taken to be non-axisymmetric, and Gram–Schmidt orthonormalization is used to remove the parasitic errors during numerical integration.

It is found that the stability characteristics are very sensitive to the velocity field in the inlet region. At all axial locations investigated the critical frequency and critical wavenumber for the Hornbeck profile are larger than the corresponding values for the Sparrow profile while the critical Reynolds number is smaller. The minimum critical Reynolds number for the Hornbeck profile is only 13250 and occurs at $\bar{X} = 0{\cdot}0032$ compared with 19780 at $\bar{X} = 0{\cdot}0049$ for the Sparrow profile. The maximum difference between the two velocity profiles occurs near the boundary-layer edge but is within 5 %. Results for the Hornbeck profile are found to be closer to the experimental data of Sarpkaya (1975).

1. Introduction

The experimentally observed instability of the pipe flow is attributed either to finite disturbances (Leite 1959) or to the presence of boundary layer in the inlet region. Tatsumi (1952) was the first to study the stability of developing pipe flow. His results are, however, unreliable as was pointed out by Chen & Sparrow (1967) and by Chen (1969). While Huang & Chen (1974*a*, *b*) did provide temporal stability results for this problem, they used the Sparrow profile, the velocity profile obtained by Sparrow *et al.* (1964), and their results do not compare well with the experimental data of Sarpkaya (1975). The reason for this discrepancy may be the fact that the Sparrow profile is not an accurate description of the developing velocity field in the pipe. The work of Schmidt & Zeldin (1969), Crane & Burley (1976), and of Shah (1978) reveals that the Hornbeck profile found by Hornbeck (1963) gives more accurate velocity description, length of entrance region and pressure drop in the developing region.

It is also well known that Squire's theorem (Squire 1933) is not applicable in the case of axisymmetric flows (Spielberg & Timan 1960). In fact the Hagen–Poiseuille flow is known to be less stable to non-axisymmetric disturbances with angular wavenumber equal to unity than to the axisymmetric disturbances (Garg & Rouleau 1972; Salwen & Grosch 1972). We therefore consider the linear spatial stability of the Hornbeck profile in the developing region of a pipe to non-axisymmetric disturbances.

2. Analysis

We can consider the developing pipe flow as nearly parallel since the radial component of velocity, V, in the entrance region is very small compared with the axial component, U, so that $U \simeq U(r)$, $V \simeq 0$ at any axial location. Each dimensionless component, say $\xi(r,\theta,x,t)$, of the Fourier series corresponding to the arbitrary infinitesimal disturbance is assumed to be of the form

$$\xi(r,\theta,x,t) = \mathscr{R}[\phi(r)\exp(ikx+in\theta-i\omega t)],$$

where $\mathscr{R}$ denotes the real part of a complex function; r, θ, x and t are the dimensionless radial, angular, axial and time co-ordinates, ω is the real frequency, n the angular wavenumber, and k the axial complex wavenumber. Substitution of such expressions for the disturbance pressure and velocity components into the linearized equations of motion leads to the following set of four coupled, linear, ordinary differential equations after some algebraic manipulation:

$$\left.\begin{aligned}
&\left(D+\frac{1}{r}\right)(f+g)+\frac{n}{r}(f-g)+2iku = 0,\\
&\left(D^2+\frac{1}{r}D-\frac{1}{r^2}-A\right)g+\frac{2n}{r^2}g-R\left(D+\frac{n}{r}\right)p = 0,\\
&\left(D^2+\frac{1}{r}D-A\right)u-R\frac{f+g}{2}DU-ikRp = 0,\\
&2RDp+\frac{n}{r}\left(D+\frac{1}{r}\right)(f-g)+A(f+g)+2ikDu = 0,
\end{aligned}\right\}\tag{2.1}$$

where

$$f(r) = v(r)+iw(r), \quad g(r) = v(r)-iw(r),$$

$$A = \frac{n^2}{r^2}+k^2+iR(kU-\omega),$$

$$D \equiv d/dr, \quad R = U_a a/\nu,$$

R is the Reynolds number, U_a is the average velocity of the basic flow at any cross-section, a is the pipe radius, p, u, v and w are the dimensionless complex eigenfunctions for disturbance pressure and for disturbance velocity components in the axial, radial and angular directions respectively, and ν is the kinematic viscosity of the fluid.

The boundary conditions require that (Garg & Rouleau 1972)

$$\left.\begin{aligned}
f(0) = u(0) = p(0) = 0,\\
f(1) = g(1) = u(1) = 0.
\end{aligned}\right\}\tag{2.2}$$

At each axial location specified by $\bar{X} = x/R$, $U(r)$ is known (Hornbeck 1963) and the solution of equations (2.1) and (2.2) yields the complex eigenvalue k for given values of n, ω and R. The flow is unstable when the disturbance grows in the downstream direction, i.e. when $k_i < 0$ where k_i is the imaginary part of k, k_r being the real part.

3. Solution

Knowing that the Hagen–Poiseuille flow is least stable to the disturbance with $n = 1$, we consider the solution of equations (2.1) and (2.2) for $n = 1$ only, hoping that the developing pipe flow may be most unstable to such a disturbance.

Let $\mathbf{Y}$ and $\mathbf{F}$ be vectors such that $\mathbf{Y} \equiv \{y_1, y_2, y_3, y_4, y_5, y_6\} \equiv \{f, g, u, Dg, Du, p\}$, and $\mathbf{F} \equiv D\mathbf{Y}$. We then find from the boundary conditions at the pipe axis ($r = 0$) and from equations (2.1) at $r = 0$ that for $n = 1$

$$\left.\begin{aligned} \mathbf{Y}(0) &= \{0, c_1, 0, 0, c_2, 0\}, \\ \mathbf{F}(0) &= \{0, 0, c_2, c_3 + Bc_1, 0, c_3\}, \end{aligned}\right\} \tag{3.1}$$

where $B = \frac{1}{2}[k^2 + iR(kU(0) - \omega)]$, and c_1, c_2 and c_3 are finite.

Thus we can use any standard marching technique, such as the fourth-order Runge–Kutta method, for integrating (2.1) from the pipe axis to the wall where satisfaction of the no-slip condition in (2.2) yields the eigenvalue k. The Gram–Schmidt orthonormalization procedure (Garg 1980) is used selectively to remove the parasitic errors during numerical integration. Convergence to the eigenvalue is achieved by Muller's method (Muller 1956). The axial spacing, $\bar{X}$, required in the Hornbeck method, was taken to be 10^{-5} for $0 < \bar{X} \leqslant 0{\cdot}01$. This was decided on the basis of our efforts to reproduce the velocity profile given in Hornbeck (1963). By numerical experimentation following Collatz (1966, p. 51) it was found that a step size of 0·0025 gives an error of $O(10^{-6})$ in the eigenvalues when computation is done in double precision mode on DEC 1090. Iteration to the neutral point was terminated for $|k_i| \leqslant 10^{-6}$.

4. Developing flow velocity profiles

While Hornbeck (1963) used the finite-difference method for solving the boundary-layer type of equations for the developing flow in a pipe, Sparrow *et al.* (1964) linearized the non-linear inertial terms in the Navier–Stokes equations to get a series solution for the velocity profile. Figure 1 shows the velocity and its gradient as obtained by the two methods at $\bar{X} = 0{\cdot}0014$ and 0·00616. It is observed that the two velocity profiles differ more from each other near the boundary-layer edge but the maximum difference is within 5 %. The velocity gradients differ marginally near the pipe wall but considerably (by as much as 100 %) near the boundary-layer edge. The velocity gradient for the Hornbeck profile is less than that for the Sparrow profile near the pipe wall but rises above it mid-way through the boundary-layer thickness. Figure 2 exhibits the development of the velocity field as the fluid moves downstream.

5. Results

As pointed out already, results for the spatial stability of the Hornbeck profile in a pipe to the non-axisymmetric disturbance with $n = 1$ are provided at several axial locations. These are compared with those available in Gupta & Garg (1981) for the Hornbeck and Sparrow profiles for an axisymmetric disturbance ($n = 0$), and in Huang (1973) for the Sparrow profile for $n = 1$.

Figures 3 and 4 show the neutral curves (ω *vs.* R and k_r *vs.* R) at several axial locations

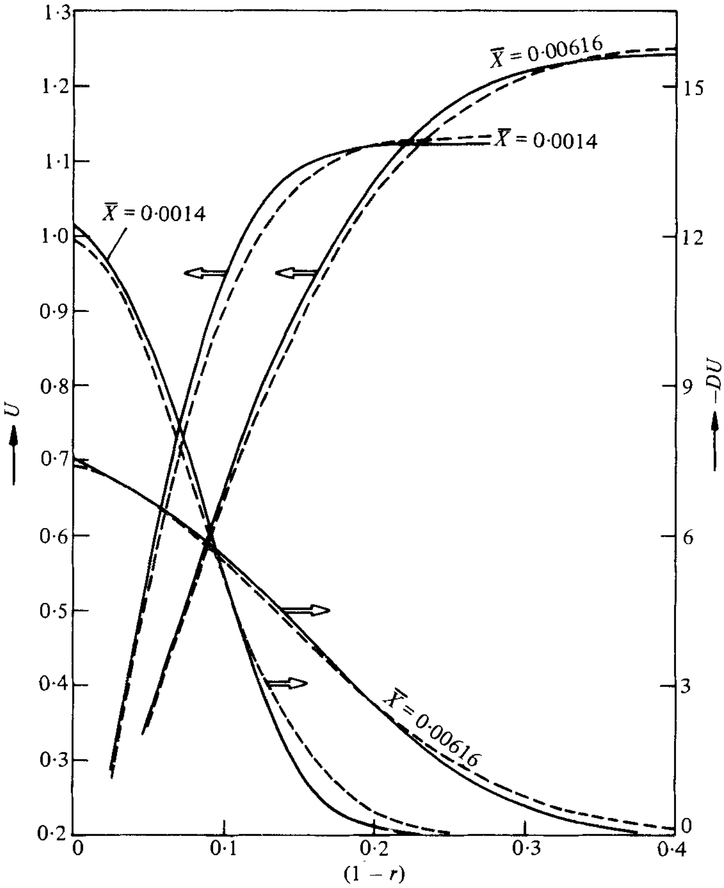

FIGURE 1. Developing flow velocity profile and its gradient in a pipe at two axial locations. ——, Hornbeck's profile; – – –, Sparrow's profile.

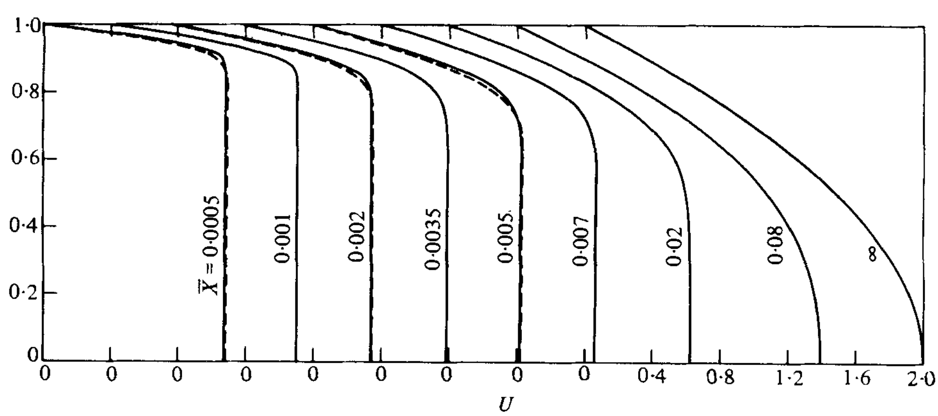

FIGURE 2. Velocity profiles at several axial locations in a pipe. ——, Hornbeck's profiles; – – –, Sparrow's profiles.

X^*	$\bar{X}$	X^*	$\bar{X}$
0·0	0·0	0·001	0·00040
0·002	0·00087	0·003	0·00140
0·004	0·00197	0·005	0·00258
0·006	0·00323	0·007	0·00392
0·008	0·00464	0·009	0·00539
0·010	0·00616	0·015	0·01043

TABLE 1. Correspondence between X^* and $\bar{X}$.

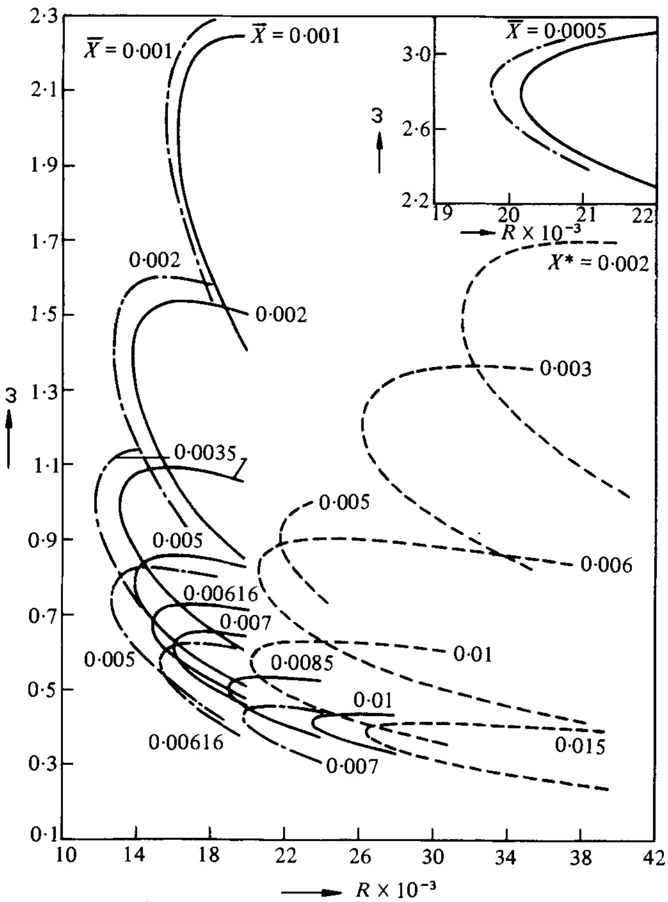

FIGURE 3. Neutral curves (ω *vs.* R) at various axial locations for Hornbeck's profile (——, $n = 1$; —·—, $n = 0$) and for Sparrow's profile (– – –, $n = 1$).

for both the Hornbeck and Sparrow profiles for $n = 0$ and 1. Correspondence between the physical co-ordinate $\bar{X}$ and the stretched co-ordinate X^* (of Sparrow *et al.* 1964) is given in the table 1. Comparison of the neutral curves corresponding to $n = 1$ for both the velocity profiles at $\bar{X} = 0{\cdot}00616$ ($X^* = 0{\cdot}01$) shows that the Hornbeck profile is unstable at much lower Reynolds numbers and over a wider frequency and wave-number range than the Sparrow profile. Similiar results were obtained by Gupta & Garg (1981) for the axisymmetric disturbance. These figures also show that in the

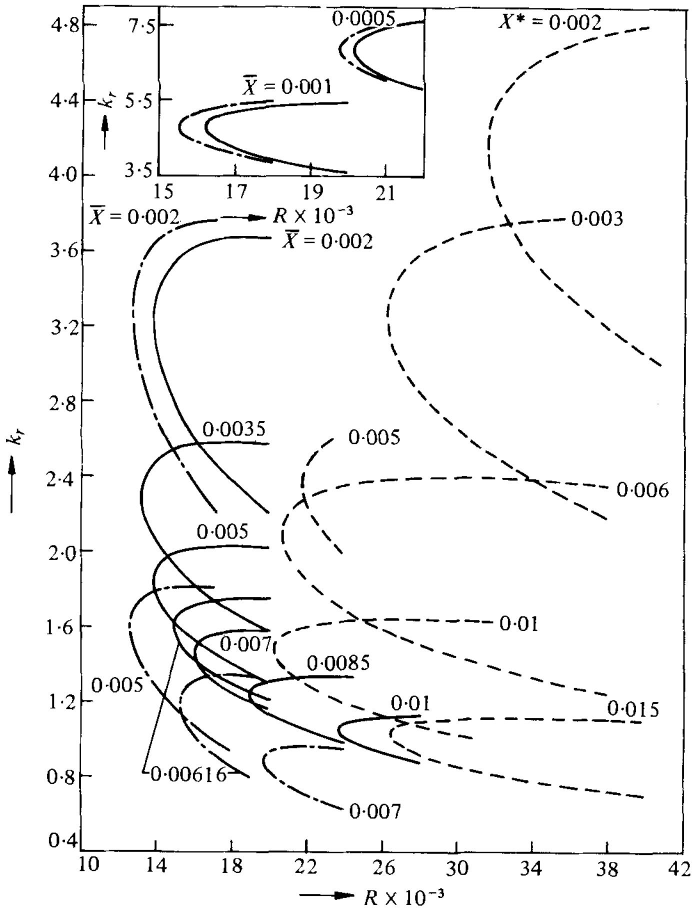

FIGURE 4. Neutral curves (k_r *vs.* R) at various axial locations for Hornbeck's profile (——, $n = 1$; —·—, $n = 0$) and for Sparrow's profile (- - -, $n = 1$).

near entry region, the Hornbeck profile is more unstable to axisymmetric disturbances than to the non-axisymmetric disturbances with $n = 1$.

Figure 5 shows the variation of critical Reynolds number R_c, critical frequency ω_c, and critical wavenumber k_{r_c} with $\bar{X}$ for both the profiles for $n = 1$. It also shows the variation of R_c with $\bar{X}$ for $n = 0$ for both the profiles and contains experimental data for R_c and ω_c as obtained by Sarpkaya (1975) for $n = 1$ disturbance. A comparison of the $R_c - \bar{X}$ curves for $n = 0$ and 1 shows that the developing pipe flow has a lower critical Reynolds number in the near entry region for an axisymmetric disturbance. While the Hornbeck profile has a lower R_c for $n = 0$ than that for $n = 1$ disturbance for $\bar{X} < 0{\cdot}006$, the Sparrow profile shows this behaviour up to only $\bar{X} = 0{\cdot}0038$. That an axisymmetric disturbance is more unstable than the non-axisymmetric disturbance with $n = 1$ in the near entry region of a pipe is in direct contrast to the well-known result for the Hagen–Poiseuille flow. It may however be explained on the basis that in the near entry region the basic flow is of boundary layer type for which Squire's theorem does hold. The minimum critical Reynolds numbers for the Hornbeck profile are 11700 and 13250 at $\bar{X} = 0{\cdot}00335$ and $0{\cdot}0032$ respectively for the axisymmetric and

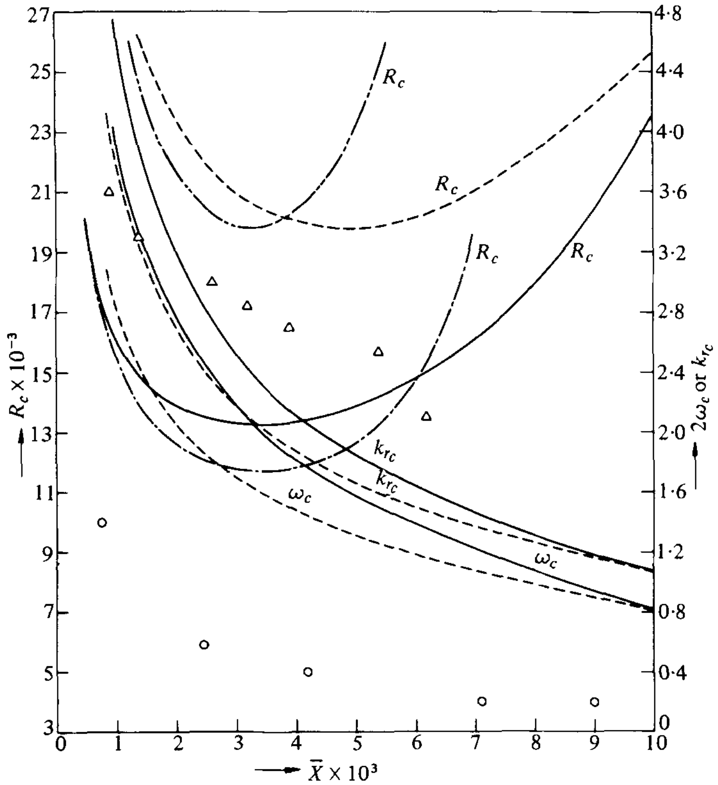

FIGURE 5. Variation of R_c, ω_c and k_{r_c} with $\bar{X}$ for Hornbeck's profile (——, $n = 1$; —·—, $n = 0$), and for Sparrow's profile (– – –, $n = 1$; —··—, $n = 0$). Experimental data for $n = 1$ (○, R_c; △, ω_c).

non-axisymmetric disturbance with $n = 1$, while for the Sparrow profile, they are 19800 and 19780 at $\bar{X} = 0{\cdot}00325$ and $0{\cdot}0049$ respectively.

As $\bar{X}$ increases the critical Reynolds number passes through a minima for both the velocity profiles. We also note from figure 5 that experimental values obtained by Sarpkaya (1975) for R_c, though well below the R_c *vs.* $\bar{X}$ curve for the Hornbeck profile, are still closer to it than to the R_c *vs.* $\bar{X}$ curve for the Sparrow profile. Sarpkaya found the minimum R_c to be about 3800 for $\bar{X}$ in the range $0{\cdot}012 \leqslant \bar{X} \leqslant 0{\cdot}02$. However, as Sarpkaya himself noted, his critical Reynolds numbers may be low due to a higher initial disturbance level than that warranted by the linear theory. Sarpkaya's experimental results for the critical frequency corresponding to the $n = 1$ disturbance also compare better with those found here for the Hornbeck profile than those found by Huang (1973) for the Sparrow profile. It is clear from figure 5 that both ω_c and k_{r_c} for the Hornbeck profile are larger than the corresponding values for the Sparrow profile, and that with increasing $\bar{X}$ both decrease first sharply and then gradually. Also, as $\bar{X}$ increases, all corresponding curves for both the velocity profiles approach each other. This is because the difference between the two velocity profiles decreases with increasing $\bar{X}$.

As already noted the maximum difference between the two velocity profiles is within 5 %. Sarpkaya (1975) points out that his experimentally observed velocity profile also agrees with the Sparrow profile to within 5 %. It would have been interesting to compare the experimental and the Hornbeck velocity profiles especially when we find that the Hornbeck velocity profile gives stability characteristics that are closer to the experimental ones. Unfortunately, however, Sarpkaya's velocity profiles are not available.

6. Conclusions

It is found that in the near entry region the developing pipe flow is more unstable to an axisymmetric disturbance than to the non-axisymmetric disturbance with $n = 1$. This is in direct contrast to the well-known result for the Hagen–Poiseuille flow. A comparison of the stability characteristics for the Hornbeck and Sparrow velocity profiles reveals that the Hornbeck profile yields much lower critical Reynolds numbers but slightly higher critical frequencies and wavenumbers than the Sparrow profile. Also, results obtained here for the Hornbeck profile are closer to the experimental values.

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